Introduction To Derivatives
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1 Derivative

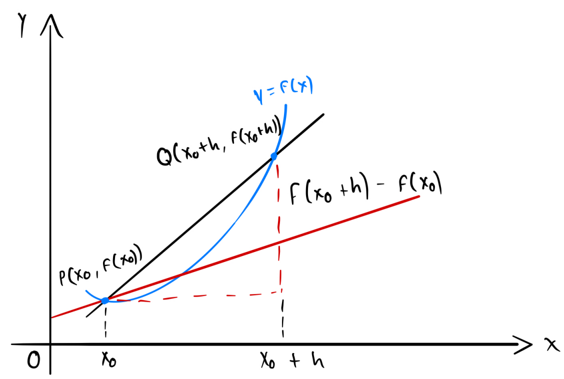
We begin this chapter by introducing a crucial definition to the understanding of derivatives represented by *figure 1*. It is very similar to the average rate of change formula covered in the **introduction to limits** packet. The main difference however, is that now we will take the limit as h (our displacement) approaches 0. Lets think on what that means. If we look at figure 1 we have x_0 and $x_0 + h$. we also have a displacement from the first point P to the point Q represented by h. If we were using the **average rate of change** formula we would be capable to connect the two points with a secant line and through rise over run find their average rate of change. In here however, because we take the limit as $h \to 0$ the distance from point P and point Q becomes so small that point Q is at the exact same spot as point P. Therefore the use of a tangent line be sufficient to find the **instantaneous rate of change**. As we progress through this chapter we will show that this definition allows us to calculate limits of functions in a matter of seconds. This is because the derivative of a function allows us to know the average rate of change at any point of $f(x)$. For now however, we will provide the proper definitions. (This brief introduction covers definitions 1 to 4).

1.1 Definition 1, Finding a Tangent Line to the Graph of a Function

The **slope of the curve** $y = f(x)$ at the point $P(x_0,f(x_0))$ is the number

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \text{ (provided the limit exists)}$$

The **tangent** line to the curve P is the line through P with this slope.



1.2 Definition 2, Rate of Change: Derivative at a Point

The derivative of a function f at a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

provided that the limit exists.

1.3 Definition 3, The Derivative

The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

provided that the limit exists.

1.4 Theorem 1, Differentiability Implies Continuity

If f has a derivative at $x = c$ then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x \to c} f(x) = f(c)$, or equivalently, that $\lim_{h \to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$f(c+h) = f(c) + (f(c+h) - f(c)) \text{ (add and subtract } f(c))$$

$$= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h \text{ (Divide and multiply by h)}$$

Now take limits as $h \to 0$. By Theorem 1 of section 2.2

$$\lim_{h \to 0} f(c+h) = \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h \to 0}$$

$$= f(c) + f'(c) \cdot 0$$

$$= f(c) + 0$$

$$= f(c)$$

.

1.5 Definition 4, Instantaneous rate of change

The Instantaneous rate of change of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

1.6 Derivative Rules

1.6.1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

1.6.2 Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx} x^n = n x^{n-1}.$$

1.6.3 Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

1.6.4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

1.6.5 Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

1.6.6 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\tfrac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v.$$

1.6.7 Derivative Quotient Rule

IF u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{du}{dx}}{v^2}.$$

1.6.8 Irrational Exponents and Power Rule

For any $x > 0$ and for any real number n,

$$\frac{d}{dx}x^n = e^{nln(x)}.$$

1.6.9 Derivative of trigonometric functions

1. $\frac{d}{dx}(\sin x) = \cos x.$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\sec x) = \sec x \tan x$
5. $\frac{d}{dx}(\cot x) = -\csc^2 x$
6. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

1.6.10 Proving the derivative of sine = cosine

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h\to 0} \tfrac{f(x+h)-f(x)}{h} = f'(x) = \lim_{h\to 0} \tfrac{\sin(x+h)-\sin}{h}$$

$$= \lim_{h\to 0} = f'(x) = \frac{(\sin x \cos h + \cos x \,\text{sinh}) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

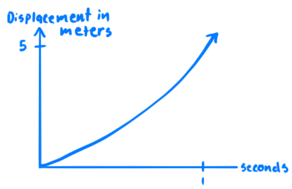
$$= \lim_{h\to 0}\left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\left(\cos x \cdot \frac{\sin h}{h}\right)$$

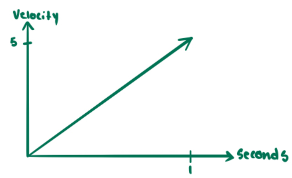
$$= \sin x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0}\frac{\sin h}{h}$$

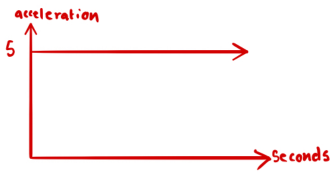
$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

1.7 Displacement, Velocity(Speed) and Acceleration

In order to better understand how displacement, velocity and acceleration are connected lets look at a real world example. Suppose that a car at a red light starts to drive. We will then analyze the cars movements from 0 to 1 seconds. In the displacement graph (notice that $f(x)$ is a curve) we show that within 1 second we moved a total of 5 meters. If we then take the derivative of our function $f(x)$ we will get a straight line for $f'(x)$ Now that we have a straight line we can easily calculate the average rate of change using the points (0,0) and (1,5). This will then give us a velocity of 5 meters per second. Lastly if we take the derivative of velocity (acceleration) our function for $f''(x)$ will become constant at the given slope of 5m/s. Notice how if we have a curved line $f(x)$ it's derivative will produce a straight line $f'(x)$ and if we take the derivative of a straight line we will get a constant $f''(x)$, furthermore, if we were to look for the third derivative (jerk) we could say that the derivative of a constant is zero.







1.8 Definition 5 - Velocity(Instantaneous Velocity)

Velocity is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

1.9 Definition 6 - Speed

Speed is the absolute value of velocity.

$$\text{Speed} = \mid v(t) \mid = \mid \frac{ds}{dt} \mid$$

1.10 Definition 7 - Acceleration

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}.$$

1.11 Definition 8 - Jerk

Jerk is the derivative of acceleration with respect to time

$$j(t) = \frac{da}{dt} = \frac{d^3}{dt^3}$$

1.12 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

1.12.1 Proving The Chain Rule

Let Δu be the change in u when x changes by Δx, so that

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

if $\Delta u \neq 0$, we can write that the fraction $\Delta y/\Delta x$ as the product

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

and take the limit as $\lim\limits_{\delta x \to 0}$:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta y}$$

1.12.2 A Simple Derivative With Chain Rule

In order to display the chain rule lets derivative the following problem $\sec x^2$. The thing we need to keep in mind is that the derivative of a chain rule is the derivative of the outside times the derivative of the inside. We solve this problem the following way;

$$\text{First notice that } \sec x^2 = (\sec x)^2$$

Now we take the derivative

$$\frac{d}{dx}\,(\sec x)^2 = 2\sec x \cdot \sec x \tan x$$

Once again, the derivative of the outside gives us $2\sec x$ and we multiply this by the derivative of the inside $\sec x \tan x$.

$$\text{A little bit of rewriting gives us that } \frac{d}{dx}(\sec x)^2 = 2\sec x^2 \tan x.$$

1.13 The Derivative Rule for Inverses - Theorem

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of (f^{-1})' at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

1.14 The Number e as a limit - Theorem

The number e can be calculated as the limit

$$e = \lim_{x \to 0}(1+x)^{\frac{1}{x}}.$$

1.14.1 Proving Theorem -

if $f(x) = \ln x$ then $f'(x) = 1/x$, so $f'(1) = 1$. But by the definition of a derivative,

$$f'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = \lim_{x \to 0} \frac{f(1+x)-f(1)}{x}$$

$$= \lim_{h \to 0} \frac{ln(1+x)-ln(1)}{x} = \lim_{x \to 0} \frac{1}{x} ln(1+x)$$

$$= \lim_{x \to 0} \ln(1+x)^{1/x} = \ln[\lim_{x \to 0}(1+x)^{1/x}].$$

Because $f'(1) = 1$, we have

$$\ln[\lim_{x \to 0}(1+x)^{1/x}] = 1.$$

Therefore, exponentiating both sides we get

$$\lim_{x \to 0}(1+x)^{1/x} = e.$$

1.15 Definitions - Inverse Trigonometric Functions

1. $y = \arctan x$ is the number in $(\frac{\pi}{2}, \frac{\pi}{2})$ for which $\tan y = x$.
2. $y = \text{arccot}\, x$ is the number in $(0,\pi)$ for which $\cot y = x$.
3. $y = \text{arcsec}\, x$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.
4. $y = \text{arccsc}\, x$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

1.16 Linearization

if f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x-a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

1.16.1 What is a Linearization

The main idea a reader should get from a linearizations is that they help make better approximations of values. We go deeper into this idea in Calculus 2 when we deal with Maclaurin and Taylor series.

1.17 Differentials

Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is

$$dy = f'(x)dx.$$

1.17.1 Example of a Differential

When we deal with differentials we can treat them similar to any derivative problem. For instance we wanted to find the differential of dy if $y = x^5 + 37x$. The only thing we need to do is take the derivative, thus;

$$\frac{d}{dx}y = \frac{d}{dx}(x^5 + 37x) \text{ is equal to } dy = (5x^4 + 37)dx.$$